

A note on the limiting entry and return times distributions for induced maps

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Abstract

For ergodic measures we consider the return and entry times for a measure preserving transformation and its induced map on a positive measure subset. We then show that the limiting entry and return times distributions are the same for the induced maps as for the map on the entire system. The only assumptions needed are ergodicity and that the measures of the sets along which the limit is taken go to zero.

1 Introduction

In [4] it was shown that on manifolds the limiting return times distributions for an ergodic map is the same for the induced map on a subset if the limit is along a sequence of metric balls. The theorem was proven for measures that satisfy the Lebesgue density theorem, which include absolutely continuous measures and also Radon measures. Here we will show a general version of that statement that only requires ergodicity and does not impose any other restrictions.

The statistics of entry and return times has been studied to quite some degree in the last two decades in particular. A number of results have been achieved for a variety of systems that have good mixing properties as for instance for Axiom A systems or subshifts of finite type and their equilibrium states by using the Laplace transform (Hirata [9], Coelho [5] and Collet) or using the moment method (Pitskel [14] and others as for instance in [7] for rational maps). Galves and Schmitt [6] showed that entry times are exponentially distributed for ϕ -mixing systems and also gave error terms by a method that later was considerably expanded and sharpened by Abadi [1] for ϕ -mixing and later even some α -mixing systems. A more elementary counting approach by Abadi and Vergne [3] proved that ϕ -mixing measures have exponentially decaying entry and return times. They also have results on multiple return times which are Poissonian for ψ -mixing measures. In [4] a reduction to induced maps was used to obtain the limiting distribution of return times. The

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important connection there was a result that establishes that the limiting return times distribution for the given system is the same as the limiting return times distribution of an induced system. This is the result we expand and sharpen in this note. It allows to obtain distribution results for many more systems because instead of checking mixing properties for the given system one can consider the jump transform which is an induced map on a suitable subset so that the induced map has good expanding properties that result in exponential decay of correlations. Such an approach has been particularly successful in the study of parabolic interval maps like the Manneville-Pommes map which is non-uniformly expanding because it has a parabolic fixed point where the derivative is equal to 1. The induced map on any interval not including the parabolic fixed point is uniformly expanding and its statistical properties can more easily be analysed exploiting the quasi compactness of the transfer operator and the consequential exponential decay of correlations.

2 Return times and the induced map

Let Ω be a measure space with σ -algebra \mathcal{B} and μ a probability measure. Moreover T is a measure preserving map on Ω . We assume that μ is ergodic. For $U \subset \Omega$ we define the function

$$\tau_U(x) = \min\{j \geq 1 : T^j x \in U\},$$

which is the *entry time* for x if $x \in \Omega$ and is the *return time* if $x \in U$. We put $\tau_U(x) = \infty$ if the forward orbit of x never enters U . Poincaré's recurrence theorem states that $\tau_U(x) < \infty$ for almost every $x \in U$ for any finite T -invariant measure μ on Ω and Kac's theorem [10] tells us that τ_U is integrable on U . In fact

$$\int_U \tau_U(x) d\mu(x) = 1$$

if $\mu(U) > 0$. Let us note that τ_U is not necessarily integrable over Ω , in fact $\int_\Omega \tau_U d\mu < \infty$ if and only if τ_U is square integrable over U .

For a subset $U \subset \Omega$, $\mu(U) > 0$, let us denote by $\hat{T} = T^{\tau_U} : U \rightarrow U$ the *induced map*. \hat{T} exists by Poincaré's (or Kac's) theorem almost everywhere. We also have the *induced measure* $\hat{\mu}$ which is defined on U by $\hat{\mu}(A) = \frac{\mu(A)}{\mu(U)}$ for all measurable $A \subset U$. The induced measure $\hat{\mu}$ is \hat{T} -invariant and ergodic (see e.g. [13]).

2.1 Entry times distributions

Let $B \subset \Omega$ ($\mu(B) > 0$) and put for (parameter values) $t > 0$

$$F_B(t) = \mathbb{P} \left(\tau_B > \frac{t}{\mu(B)} \right) = \mu \left(\left\{ x \in \Omega : \tau_B(x) > \frac{t}{\mu(B)} \right\} \right)$$

for the entry time distribution to B . The entry times distribution $F_B(t)$ is locally constant on intervals of length $\mu(B)$ and has jump discontinuities at values t which are integer multiples of $\mu(B)$. For any $s \in \mathbb{N}_0$ one has

$$\{\tau_B > s + 1\} = T^{-1}\{\tau_B > s\} \setminus T^{-1}B \quad (1)$$

and consequently

$$\mathbb{P}(\tau_B = s + 1) = \mathbb{P}(\tau_B > s) - \mathbb{P}(\tau_B > s + 1) \leq \mu(B)$$

which shows that the jumps at the discontinuities are at most $\mu(B)$. Hence

$$|F_B(t) - F_B(s)| \leq |t - s| + \mu(B)$$

for all $t, s > 0$.

Now let $B_j \subset \Omega$ ($\mu(B_j) > 0$) be a sequence of subsets so that $\mu(B_j) \rightarrow 0^+$ as $j \rightarrow \infty$. We want to assume that $F_{B_j}(t)$ converges pointwise (in t) to a limiting distribution $F(t)$ as $j \rightarrow \infty$. Let us note that the regularity (1) the limiting distribution $F(t)$ is Lipschitz continuous with Lipschitz constant 1 and consequently F is continuous.

Lacroix [12] has shown that if $F(t)$ is an eligible limiting distribution, that is it satisfies $F(0) = 1$, is continuous, convex, monotonically decreasing on $(0, \infty)$ and $F(t) \rightarrow 0^+$ as $t \rightarrow \infty$, then for any ergodic T -invariant probability measure μ there exists a sequence of positive measure sets $B_j \subset \Omega$ so that $\mu(B_j) \rightarrow 0$ and such that $F(t) = \lim_{j \rightarrow \infty} F_{B_j}(t)$ for every $t \in (0, \infty)$. The sets B_j are typically pretty wild looking and in particular won't be topological balls or cylinder sets for a given partition. A similar result was shown for return times in [11] although the two results are equivalent by [8].

For a positive measure subset $U \subset \Omega$ let us now consider the induced system $(U, \hat{T}, \hat{\mu})$ which carries the entry time function $\hat{\tau}_B(x) > \min\{j \geq 1 : \hat{T}^j x \in B\}$ for sets $B \subset U$, $\hat{\mu}(B) > 0$. As above we can then define the entry times distribution

$$\hat{F}_B(t) = \hat{\mathbb{P}}\left(\hat{\tau}_B > \frac{t}{\hat{\mu}(B)}\right) = \hat{\mu}\left(\left\{x \in U : \hat{\tau}_B(x) > \frac{t}{\hat{\mu}(B)}\right\}\right)$$

The following theorem shows that a restricted system $(U, \hat{T}, \hat{\mu})$ has the same limiting entry times distribution as the original system (Ω, T, μ) .

Theorem 1. *Let μ be ergodic, $U \subset \Omega$, $\mu(U) > 0$. Assume there exists a sequence of sets $B_j \subset U$, $\mu(B_j) \rightarrow 0^+$, so that either the limiting entry times distribution for (Ω, T, μ)*

$$F(t) = \lim_{j \rightarrow \infty} F_{B_j}(t)$$

exists, or the limiting entry times limiting distribution

$$\hat{F}(t) = \lim_{j \rightarrow \infty} \hat{F}_{B_j}(t)$$

for the induced system exists $(U, \hat{T}, \hat{\mu})$ exists.

Then both limiting entry times distributions exist and moreover $F(t) = \hat{F}(t)$ for all $t > 0$.

Proof. Let $B = B_j$. We first relate τ_B to $\hat{\tau}_B$ ($B \subset U, \mu(B) > 0$). If we put $m = \hat{\tau}_B(x)$, $x \in U$, then

$$\tau_B(x) = \tau_U(x) + \tau_U(\hat{T}x) + \tau_U(\hat{T}^2x) + \cdots + \tau_U(\hat{T}^{m-1}x) = n^m(x),$$

where we wrote the ergodic sum of the function $n = \tau_U|_U$ for the return time on (U, \hat{T}) . By the Birkhoff ergodic theorem on $(U, \hat{T}, \hat{\mu})$ we get as $\hat{\mu}$ is ergodic:

$$\frac{1}{m} \tau_B(x) = \frac{1}{m} n^m(x) \rightarrow \int_U n(x) d\mu(x) = \int_U \tau_U(x) \frac{d\mu(x)}{\mu(U)} = \frac{1}{\mu(U)}$$

as $m \rightarrow \infty$ by Kac's theorem for almost every $x \in U$.

Let $\varepsilon > 0$, then there exists $G_\varepsilon \subset U$, and $M_\varepsilon \in \mathbb{N}$ so that

$$\left| \frac{1}{m} n^m(x) - \frac{1}{\mu(U)} \right| < \varepsilon \quad \forall x \in G_\varepsilon, m \geq M_\varepsilon$$

and $\mu(G_\varepsilon^c) < \varepsilon$. Thus

$$\tau_B(x) = \sum_{j=0}^{\hat{\tau}_B(x)-1} \tau_U \circ \hat{T}^j = \frac{\hat{\tau}_B(x)}{\mu(U)} + \mathcal{O}(\hat{\tau}_B(x)\varepsilon)$$

for all $x \in G_\varepsilon$ such that $\hat{\tau}_B(x) \geq M_\varepsilon$. Since τ_U is integrable on U there exists a $\delta > 0$ (depending on ε) so that $\int_S \tau_U d\mu < \varepsilon$ for any set $S \subset U$ for which $\mu(S) < \delta$. We can assume that $\mu(G_\varepsilon^c) < \min(\delta, \varepsilon)$.

For $j = 0, 1, 2, \dots$ put $A_j = \Omega \setminus T^{-j}U = T^{-j}U^c$ and

$$D_j^k = \bigcap_{\ell=j}^k A_\ell = \{x \in \Omega : T^\ell x \notin U \forall \ell = j, \dots, k\}$$

for $0 \leq j \leq k$. Then for any $j \in \mathbb{N}$

$$\{x \in \Omega : \tau_U(x) = j\} = T^{-j}U \cap D_1^{j-1} = D_1^{j-1} \setminus D_1^j.$$

On the other hand we also have

$$\{x \in U : \tau_U(x) \geq j\} = U \cap D_1^{j-1}.$$

We now do the following decomposition (as $D_1^{j-1} = \{x \in \Omega : \tau_U(x) \geq j\}$):

$$\begin{aligned} F_B(t) &= \int_\Omega \chi_{\tau_B > s} d\mu \\ &= \sum_j \int_{\{\tau_U=j\}} \chi_{\tau_B > s} d\mu \\ &= \sum_j \left(\int_{D_1^{j-1}} \chi_{\tau_B > s} d\mu - \int_{D_1^j} \chi_{\tau_B > s} d\mu \right), \end{aligned}$$

as $D_1^j \subset D_1^{j-1}$, where we wrote $s = \frac{t}{\mu(B)}$. For the second term in the last line consider

$$\int_{D_0^{j-1}} \chi_{\tau_B > s} d\mu = \int_\Omega \left(\chi_{D_0^{j-1}} \chi_{\tau_B > s} \right) \circ T d\mu = \int_{D_1^j} \chi_{\tau_B > s} \circ T d\mu$$

as $T^{-1}D_0^{j-1} = D_1^j$. The inclusions

$$\{\tau_B > s+1\} \subset T^{-1}\{\tau_B > s\} \subset \{\tau_B > s+1\} \cup T^{-1}B$$

imply the inequalities

$$\int_{D_1^j} \chi_{\tau_B > s+1} d\mu \leq \int_{D_1^j} \chi_{\tau_B > s} \circ T d\mu \leq \int_{D_1^j} \chi_{\tau_B > s+1} d\mu + \int_{D_1^j} \chi_{T^{-1}B} d\mu$$

where the last integral is equal to zero as $T^{-1}B \cap D_1^j = T^{-1}(B \cap D_0^{j-1}) = \emptyset$ because $D_0^{j-1} \subset U^c \subset B^c$ for $j \geq 1$. Thus

$$\int_{D_1^j} \chi_{\tau_B > s} d\mu = \int_{D_1^j} \chi_{\tau_B > s-1} \circ T d\mu = \int_{D_0^{j-1}} \chi_{\tau_B > s-1} d\mu$$

and

$$\begin{aligned} F_B(t) &= \sum_j \left(\int_{D_1^{j-1}} \chi_{\tau_B > s} d\mu - \int_{D_0^{j-1}} \chi_{\tau_B > s-1} d\mu \right) \\ &= \sum_j \int_{C_j} \chi_{\tau_B > s} d\mu + E_0 \\ &= \int_U \tau_U \chi_{\tau_B > s} d\mu + E_0 \end{aligned}$$

where we put $C_j = D_1^{j-1} \setminus D_0^{j-1} = \{x \in U : \tau_U(x) \geq j\}$ and used that $\sum_j \chi_{C_j} = \tau_U$ on U (as $|\{j : x \in C_j\}| = \tau_U(x) \forall x \in U$). To estimate the error term E_0 note that $\mathbb{P}(\tau_B = s) = \mathbb{P}(\tau_B > s-1) - \mathbb{P}(\tau_B > s) \leq \mu(B)$ (see the remark preceding the theorem). Since $D_0^{j-1} = \{x \in U^c : \tau_U(x) \geq j\}$ one has

$$-E_0 = \sum_j \int_{D_0^{j-1}} \chi_{\tau_B = s} d\mu \leq \int_{\Omega} \tau_U \chi_{\tau_B = s} d\mu.$$

In order to show that E_0 goes to zero as $\mu(B)$ decreases to zero let us put $B_{k,j} = U_k \cap T^{-j}\{\tau_B = s\}$ for $j = 0, 1, \dots, k-1$ and $k = 1, 2, \dots$, where $U_k = \{x \in U : \tau_U(x) = k\}$. Then $B_{k,j} \cap B_{k,i} = \emptyset$ if $i \neq j$ because if there were an $x \in B_{k,j} \cap B_{k,i}$ it would imply that $T^j x, T^i x \in \{\tau_B = s\}$ and therefore, assuming $j > i$, one would get the contradiction $s = \tau_B(T^j x) > \tau_B(T^i x) = s$. Hence the sets $B_{k,j}$ are pairwise disjoint for $k \in \mathbb{N}$ and $j = 0, \dots, k-1$. In other words, for every $x \in B_k$ there is a unique $j \in [0, k)$ so that $T^j x \in \{\tau_B = s\}$. Consequently $\{\tau_B = s\} = \dot{\bigcup}_{k=1}^{\infty} \dot{\bigcup}_{j=0}^{k-1} T^j B_{k,j}$, and since $\mu(T^j B_{k,j}) \geq \mu(B_{k,j})$ we obtain

$$\mu(\tilde{B}) = \sum_{k=1}^{\infty} \sum_{j=0}^{k-1} \mu(B_{k,j}) \leq \sum_{k=1}^{\infty} \sum_{j=0}^{k-1} \mu(T^j B_{k,j}) = \mathbb{P}(\tau_B = s) \leq \mu(T^{-s}B) = \mu(B),$$

where $\tilde{B} = \dot{\bigcup}_{k=1}^{\infty} \dot{\bigcup}_{j=0}^{k-1} B_{k,j}$ ($\tilde{B} \subset U$). For $x \in \tilde{B}$ for which $T^j x \in \{\tau_B = s\}$ one has $\tau_U(x) \geq \tau_U(T^j x)$ and therefore

$$|E_0| \leq \int_{\tilde{B}} \tau_U d\mu < \varepsilon$$

as we can assume that $\mu(B) < \delta$.

Replacing τ_B by $\hat{\tau}_B$ using the relation $\frac{\tau_B}{\hat{\tau}_B} = \frac{1}{\mu(U)} + \mathcal{O}(\varepsilon)$ implies $\tau_B = \frac{\hat{\tau}_B}{\mu(U)} + \eta$ where $\eta : U \rightarrow \mathbb{R}$ has the bound $|\eta| \leq \varepsilon \mu(U) \hat{\tau}_B \leq \varepsilon \hat{\tau}_B$. Hence

$$F_B(t) = \int_U \tau_U \chi_{\hat{\tau}_B > \frac{t}{\mu(B)} + \eta} d\mu + E_0.$$

Now we want to introduce a power k of the induced map \hat{T} so that we can average over k and use the ergodic theorem on $(U, \hat{T}, \hat{\mu})$. By \hat{T} -invariance of $\hat{\mu}$

$$\begin{aligned} F_B(t) &= \int_U \left(\tau_U \chi_{\hat{\tau}_B > \frac{t}{\mu(B)} + \eta} \right) \circ \hat{T}^k d\mu + E_0 \\ &= \int_{G_\varepsilon^c} \left(\tau_U \chi_{\hat{\tau}_B > \frac{t}{\mu(B)} + \eta} \right) \circ \hat{T}^k d\mu + E_0 + H_k, \end{aligned}$$

where we get for the error

$$H_k = \int_{G_\varepsilon^c} \left(\tau_U \chi_{\hat{\tau}_B > \frac{t}{\mu(B)} + \eta} \right) \circ \hat{T}^k d\mu \leq \int_{G_\varepsilon^c} \tau_U \circ \hat{T}^k d\mu = \int_{\hat{T}^{-k} G_\varepsilon^c} \tau_U d\mu < \varepsilon$$

since by assumption $\mu(\hat{T}^{-k} G_\varepsilon^c) = \mu(G_\varepsilon^c) < \delta$ as μ restricted to U is \hat{T} -invariant.

In the principal term we want to exploit the identity $\sum_j \chi_{C_j} = \tau_U$ on U . For that purpose note that

$$\left\{ x \in U : \hat{\tau}_B(\hat{T}^k x) \geq s \right\} \setminus \bigcup_{\ell=1}^{k-1} \hat{T}^{-\ell} B = \{x \in U : \hat{\tau}_B(x) \geq s + k\}$$

which yields (here we use $s = \frac{t}{\mu(B)} + \eta$)

$$F_B(t) = \int_{G_\varepsilon} \left(\tau_U \chi_{\hat{\tau}_B > \frac{t}{\mu(B)} + \eta + k} \right) \circ \hat{T}^k d\mu + E_0 + H_k + K_k,$$

where the individual errors are bounded by:

$$K_k \leq \int_{G_\varepsilon} \tau_U \circ \hat{T}^k \sum_{\ell=1}^{k-1} \chi_B \circ \hat{T}^\ell d\mu.$$

We now estimate the average error over $k \in \{0, 1, \dots, n-1\}$:

$$\begin{aligned} \hat{K}_n &= \frac{1}{n} \sum_{k=0}^{n-1} K_k \\ &= \int_{G_\varepsilon} \frac{1}{n} \sum_{k=0}^{n-1} \sum_{\ell=1}^{k-1} (\tau_U \circ \hat{T}^k) (\chi_B \circ \hat{T}^\ell) d\mu \\ &= \int_{G_\varepsilon} \sum_{\ell=1}^{n-2} (\chi_B \circ \hat{T}^\ell) \left(\frac{1}{n} \sum_{k=\ell+1}^{n-1} \tau_U \circ \hat{T}^k \right) d\mu \\ &\leq c_1 \frac{1}{\mu(U)} \int_{G_\varepsilon} \sum_{\ell=1}^{n-2} (\chi_B \circ \hat{T}^\ell) d\mu \\ &\leq c_1 n \hat{\mu}(B) \end{aligned}$$

where we used the estimate

$$\frac{1}{n} \sum_{k=\ell+1}^{n-1} \tau_U \circ \hat{T}^k \leq \frac{1}{n} \sum_{k=0}^{n-1} \tau_U \circ \hat{T}^k \leq \frac{1}{\mu(U)} + \varepsilon \leq c_1 \frac{1}{\mu(U)}$$

for some constant c_1 and for all $x \in G_\varepsilon$ provided $n \geq M_\varepsilon$. Thus

$$F_B(t) = \frac{1}{n} \sum_{k=1}^n \int_{G_\varepsilon} \left(\tau_U \circ \hat{T}^k \right) \chi_{\hat{\tau}_B > \frac{1}{1-\eta'} \frac{t+k\hat{\mu}(B)}{\hat{\mu}(B)}} d\mu + E_0 + \hat{H}_n + \hat{K}_n,$$

where $\hat{H}_n = \frac{1}{n} \sum_{k=0}^{n-1} H_k < \varepsilon$ and $\eta'' : U \rightarrow \mathbb{R}$ satisfies the bound $|\eta''| < \varepsilon$. Consequently (as $|E_0 + \hat{K}_n| < \varepsilon + c_1 n \hat{\mu}(B)$)

$$\begin{aligned} F_B(t) &= \int_{G_\varepsilon} \frac{1}{n} \sum_{k=1}^n \left(\tau_U \circ \hat{T}^k \right) \chi_{\hat{\tau}_B > (1+\eta'') \frac{t}{\hat{\mu}(B)}} d\mu + \mathcal{O}(\varepsilon + n \hat{\mu}(B)) \\ &= \int_{G_\varepsilon} \chi_{\hat{\tau}_B > (1+\eta'') \frac{t}{\hat{\mu}(B)}} d\hat{\mu} + \mathcal{O}(\varepsilon + n \hat{\mu}(B)) \end{aligned}$$

as $\frac{1}{n} \sum_{k=1}^n \tau_U \circ \hat{T}^k = \frac{1}{\mu(U)} + \mathcal{O}(\varepsilon)$ on G_ε , where $\eta'' : U \rightarrow \mathbb{R}$ satisfies $|\eta''| < c_2 |\eta'| + \frac{n}{t} \hat{\mu}(B)$ ($c_2 > 0$). To adjust for the ‘time shift’ in the lower bound of the entry function, we use the fact that $|\hat{F}_B(t) - \hat{F}_B(s)| \leq |t-s| + \hat{\mu}(B)$ and thus obtain (for a c_3)

$$|F_B(t) - \hat{F}_B(t)| < c_3 \varepsilon + \left(\frac{n}{t} + 1 \right) \hat{\mu}(B) + c_1 n \hat{\mu}(B)$$

for all $n \geq M_\varepsilon$. If $\mu(B_j)$ is small enough so that $\hat{\mu}(B_j) < \min(\frac{\varepsilon t}{M_\varepsilon}, \frac{\varepsilon}{(c_1+1)n})$ (if we choose $n = M_\varepsilon$ this requires $\hat{\mu}(B_j) < \frac{\varepsilon}{M_\varepsilon} \min(t, \frac{1}{c_1+1})$) then

$$|F_{B_j}(t) - \hat{F}_{B_j}(t)| < (c_3 + 1) \varepsilon$$

and as $\mu(B_j) \rightarrow 0$ ($j \rightarrow \infty$) we obtain $|F(t) - \hat{F}(t)| < (c_3 + 1) \varepsilon$ for any positive ε . Thus if the limiting distribution $F(t)$ exists then also the limiting distribution $\hat{F}(t)$ exists and vice versa. Moreover we obtain equality: $F(t) = \hat{F}(t)$ for all $t > 0$. \blacksquare

2.2 Return times distributions

The restriction of the function τ_B to the set $B \subset \Omega$ is called the *return time function* and we correspondingly call

$$\tilde{F}_B(t) = \mathbb{P}_B \left(\tau_B > \frac{t}{\mu(B)} \right)$$

the *return times distribution*. For instance, if Ω is the shiftspace Σ and $B = U(x_0 x_1 \cdots x_{n-1})$ is an n -cylinder then $\tau_B(\vec{x})$ for $\vec{x} \in B$ measures the ‘time’ it takes to see the word $x_0 x_1 \cdots x_{n-1}$ again, that is

$$\tau_B(\vec{x}) = \min\{j \geq 1 : x_j x_{j+1} \cdots x_{j+n-1} = x_0 x_1 \cdots x_{n-1}\}.$$

The function $\tilde{F}_B(t)$ then measures the probability to see the first n -word again after rescaled time $t/\mu(B)$.

Similarly for the induced system $(U, \hat{T}, \hat{\mu})$ we have the return times distribution

$$\hat{\tilde{F}}_B(t) = \hat{\mathbb{P}}_B \left(\hat{\tau}_B > \frac{t}{\hat{\mu}(B)} \right) = \hat{\mu} \left(\left\{ x \in B : \hat{\tau}_B(x) > \frac{t}{\hat{\mu}(B)} \right\} \right).$$

In order to get a similar result on the relation between return times for the original system and the induced system, we will need the following result.

Proposition 2. [8] *Let $B_j \subset \Omega$ ($\mu(B_j) > 0$) be a sequence of sets so that $\mu(B_j) \rightarrow 0^+$. If one of the limits $F(t) = \lim_{j \rightarrow \infty} F_{B_j}(t)$, $\tilde{F}(t) = \lim_{j \rightarrow \infty} \tilde{F}_{B_j}(t)$ exists (point-wise) then so does the other limit and moreover*

$$F(t) = \int_t^\infty \tilde{F}(s) ds.$$

While the limiting entry times distribution $F(t)$ is always Lipschitz continuous with Lipschitz constant 1, the same does not apply to the limiting return times distribution $\tilde{F}(t)$ which in fact can have (at most countable many) discontinuities. In particular, if the sets B_j contract to a periodic point, then $\tilde{F}(t)$ will have a discontinuity at $t = 0$ with $\lim_{t \rightarrow 0^+} \tilde{F}(t) < 1$. Also note that since the limiting entry distribution F is Lipschitz continuous the limiting return distribution $\tilde{F}(t)$ is monotonically decreasing to zero which implies that $F(t)$ is in fact always convex.

One consequence of this result is that the limiting entry times distribution and return times distribution are the same only if they are exponential, that is $\tilde{F} = F$ if and only if $F(t) = \tilde{F}(t) = e^{-t}$. We use this proposition to obtain the corresponding result of Theorem 1 for the limiting return times distribution.

Theorem 3. *Let μ be ergodic, $U \subset \Omega$, $\mu(U) > 0$. Assume there exists a sequence of sets $B_j \subset U$, $\mu(B_j) \rightarrow 0^+$, so that one of the two limiting return times distribution*

$$\text{either } \tilde{F}(t) = \lim_{j \rightarrow \infty} \tilde{F}_{B_j}(t), \quad \text{or} \quad \hat{\tilde{F}}(t) = \lim_{j \rightarrow \infty} \hat{\tilde{F}}_{B_j}(t)$$

exists.

Then both limiting return times distributions exist and moreover at every point of continuity $t \in \mathbb{R}^+$ one has equality $\tilde{F}(t) = \hat{\tilde{F}}(t)$.

This is the result that was proven in [4] in 2003 for Radon measures on Riemann manifolds using the Lebesgue Density theorem. The limit there was along metric balls B_j that shrink to a point x and with the implication that the existence of the limiting return times distribution in the induced system (plus the non-degeneracy condition $\tilde{F}(0^+) = 1$) implies the limiting return times distribution for the entire system and that the two limiting distributions are equal at points of continuity.

Proof of Theorem 3. Assume that, say, the limit $\tilde{F}(t) = \lim_{j \rightarrow \infty} \tilde{F}_{B_j}(t)$ exists. By Proposition 2 this implies the also the limiting distribution $F(t) = \lim_{j \rightarrow \infty} F_{B_j}(t)$ exists. By Theorem 1 we get that the limit $\hat{\tilde{F}}(t) = \lim_{j \rightarrow \infty} \hat{\tilde{F}}_{B_j}(t)$ exists and satisfies

$\hat{F} = F$. Again by Proposition 2 this implies the limit $\tilde{F}(t) = \lim_{j \rightarrow \infty} \tilde{F}_{B_j}(t)$ exists. Thus, since

$$\int_t^\infty \tilde{F}(s) d\mu(s) = F(t) = \hat{F}(t) = \int_t^\infty \hat{\tilde{F}}(s) d\mu(s)$$

for all $t > 0$ we conclude that $\tilde{F}(t) = \hat{\tilde{F}}(t)$ at all points t of continuity.

Similarly one shows that the limit $\tilde{F}(t) = \lim_{j \rightarrow \infty} \tilde{F}_{B_j}(t)$ implies the return times limiting distribution $\tilde{F}(t) = \lim_{j \rightarrow \infty} \tilde{F}_{B_j}(t)$ for the whole system and also equality of the limiting distributions $\tilde{F}(t) = \hat{\tilde{F}}(t)$ at points of continuity. \blacksquare

Let us note that since the requirements in Theorem 1 and 2 are the existence of the limits they apply in particular also to examples of Lacroix and Kupsa [12, 11] where for any ergodic transformation they produce a sequence B_j that realises an arbitrary given (eligible) limiting distribution.

2.3 Example

Here we give an example where it is easy to find the limiting entry/return times distributions for the induced map. We consider the shift space $\Omega = \mathbb{N}^{\mathbb{Z}}$ with the shift transformation σ . To define the invariant measure μ we give on the state space \mathbb{N} the transition probabilities: Let $p_i \in (0, 1), i = 1, 2, \dots$, be a sequence, then we allow for the transition $i \rightarrow i + 1$ with probability p_i and for the transition $i \rightarrow 1$ with probability $q_i = 1 - p_i$. In other words, we can define a stochastic matrix M by

$$\begin{cases} M_{j,1} &= q_j \\ M_{j,j+1} &= p_j \\ M_{j,k} &= 0 \text{ otherwise, i.e. if } k \neq 1 \text{ or } k \neq j+1 \end{cases},$$

where the transition probability of the transition $j \rightarrow k$ is given by the entry $M_{j,k}$. Then $M\mathbf{1} = \mathbf{1}$ as $\sum_{k=1}^\infty M_{j,k} = M_{j,1} + M_{j,j+1} = q_j + p_j = 1 \forall j$ and M has the left eigenvector $\vec{x} = (x_1, x_2, \dots)$ (for the dominant eigenvalue 1) which satisfies

$$\begin{aligned} q_1 x_1 + q_2 x_2 + q_3 x_3 + \dots &= x_1 \\ x_j p_j &= x_{j+1} \text{ for } j = 1, 2, \dots \end{aligned}$$

One sees that the components of the left eigenvector are $x_j = x_1 P_j, j = 2, 3, \dots$, where $P_j = \prod_{i=1}^{j-1} p_i$ ($P_1 = 1$) and x_1 is chosen to make \vec{x} a probability vector ($x_1^{-1} = \sum_j P_j$). We assume $x_1 > 0$. The first equation above is satisfied as $\sum_j q_j x_j = x_1 \sum_j (P_j - P_{j+1}) = x_1$ if $P_j \rightarrow 0$ as $j \rightarrow \infty$. In this way we obtain a shift invariant probability measure μ on Ω which is ergodic as one can go from any state i to any other state j with positive probability.

Put $A_j = \{\vec{\omega} \in \Omega : \omega_0 = j\}, j = 1, 2, \dots$, and let $U = A_1$ be the return set with return/entry time function τ_U . If we put $A_{j,k} = A_j \cap \{\tau_U = k\}$ then $\vec{\omega} \in A_{j,k}$ is of the form $\omega_0 \omega_1 \dots \omega_k = j(j+1)(j+2) \dots (j+k-2)(j+k-1)1$ (symbol sequence of length $k+1$). One has

$$\mu(A_{j,k}) = \mu(A_j) p_j p_{j+1} \dots p_{j+k-2} q_{j+k-1} = x_1 P_j \frac{P_{j+k-1}}{P_j} q_{j+k-1} = x_1 P_{j+k-1} q_{j+k-1}$$

as $\mu(A_j) = x_j = x_1 P_j$. Let \mathcal{D} be the countably infinite partition of U whose partition elements are $D_j = \{\omega \in U : \tau_U(\omega) = j\}$ ($D_j = A_{1,j}$). The induced map $\hat{\sigma} : U \rightarrow U$ is a Bernoulli shift on $\hat{\Omega} = \mathcal{D}^{\mathbb{Z}}$ and the induced measure $\hat{\mu}$ is the Bernoulli measure with weights $\hat{\mu}(D_j) = \frac{1}{\mu(U)} x_1 q_j P_j$, where $\mu(U) = \sum_j x_1 q_j P_j$. If we denote by B_n the n -cylinder which contains a given point $\hat{\omega} \in \hat{\Omega}$ then the entry times $\hat{F}_{B_n}(t)$ converge to the exponential distribution e^{-t} as $n \rightarrow \infty$ for almost every $\hat{\omega}$. Hence we conclude that entry times F_{B_n} for the map σ on Ω also converge to the limiting distribution e^{-t} almost surely.

Remark. Kac's theorem states that the return time function τ_U is integrable over U and also gives the value of the integral. We can use this example to achieve that τ_U is not integrable over the entire space Ω although the measure is ergodic. The integral of τ_U over the entire space is

$$\int_{\Omega} \tau_U d\mu = \sum_{j,k} k \mu(A_{j,k}) = \sum_{j,k} k x_1 P_{j+k-1} q_{j+k-1}.$$

If we choose $p_i = \left(\frac{i}{i+1}\right)^{\alpha}$ for some $\alpha \in (1, 2)$ then $P_j = \prod_{i=1}^{j-1} \left(\frac{i}{i+1}\right)^{\alpha} = \frac{1}{j^{\alpha}}$ and since the P_j are summable, $x_1 = \left(\sum_j P_j\right)^{-1}$ is well defined and positive. Then

$$\begin{aligned} \int_{\Omega} \tau_U d\mu &= x_1 \sum_k k \sum_j \frac{1}{(j+k-1)^{\alpha}} q_{j+k-1} \\ &\geq c_1 x_1 \sum_k k \sum_j \frac{1}{(j+k-1)^{\alpha+1}} \\ &\geq c_2 \sum_k \frac{k}{k^{\alpha}} = \infty, \end{aligned}$$

as $\alpha < 2$, where we used that $q_{j+k-1} = 1 - \left(1 - \frac{1}{j+k-1}\right)^{\alpha} \geq c_1 \frac{1}{j+k-1}$ for some $c_1 > 0$. We thus see that the integral of τ_U over the entire space Ω diverges.

This can be converted to an example on a two-state shiftspace $\Sigma \subset \{0, 1\}^{\mathbb{Z}}$ by the single element mapping $\pi : \Omega \rightarrow \Sigma$ which maps $\pi(1) = 1$ and collapses all other symbols to 0, i.e. $\pi(j) = 0, j = 2, 3, \dots$. The measure μ is sent to the probability measure $\nu = \pi^* \mu$ which is invariant under the shift map.

In fact $\int_{\Omega} \tau_U d\mu$ is finite if and only if $\int_U \tau_U^2 d\mu$ is finite. So the above example is an example where the return time to U is not square integrable over U .

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